

# UNIFORMLY 2-ABSORBING PRIMARY IDEALS OF COMMUTATIVE RINGS

HOJJAT MOSTAFANASAB, ÜNSAL TEKİR AND GÜLŞEN ULUCAK

**ABSTRACT.** In this study, we introduce the concept of “uniformly 2-absorbing primary ideals” of commutative rings, which imposes a certain boundedness condition on the usual notion of 2-absorbing primary ideals of commutative rings. Then we investigate some properties of uniformly 2-absorbing primary ideals of commutative rings with examples. Also, we investigate a specific kind of uniformly 2-absorbing primary ideals by the name of “special 2-absorbing primary ideals”.

## 1. INTRODUCTION

Throughout this paper, we assume that all rings are commutative with  $1 \neq 0$ . Let  $R$  be a commutative ring. An ideal  $I$  of  $R$  is a *proper ideal* if  $I \neq R$ . Then  $Z_I(R) = \{r \in R \mid rs \in I \text{ for some } s \in R \setminus I\}$  for a proper ideal  $I$  of  $R$ . Additively, if  $I$  is an ideal of commutative ring  $R$ , then *the radical of  $I$*  is given by  $\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}$ . Let  $I, J$  be two ideals of  $R$ . We will denote by  $(I :_R J)$ , the set of all  $r \in R$  such that  $rJ \subseteq I$ .

Cox and Hetzel have introduced uniformly primary ideals of a commutative ring with nonzero identity in [6]. They said that a proper ideal  $Q$  of the commutative ring  $R$  is *uniformly primary* if there exists a positive integer  $n$  such that whenever  $r, s \in R$  satisfy  $rs \in Q$  and  $r \notin Q$ , then  $s^n \in Q$ . A uniformly primary ideal  $Q$  has order  $N$  and write  $\text{ord}_R(Q) = N$ , or simply  $\text{ord}(Q) = N$  if the ring  $R$  is understood, if  $N$  is the smallest positive integer for which the aforementioned property holds.

Badawi [3] said that a proper ideal  $I$  of  $R$  is a *2-absorbing ideal of  $R$*  if whenever  $a, b, c \in R$  and  $abc \in I$ , then  $ab \in I$  or  $ac \in I$  or  $bc \in I$ . He proved that  $I$  is a 2-absorbing ideal of  $R$  if and only if whenever  $I_1, I_2, I_3$  are ideals of  $R$  with  $I_1 I_2 I_3 \subseteq I$ , then  $I_1 I_2 \subseteq I$  or  $I_1 I_3 \subseteq I$  or  $I_2 I_3 \subseteq I$ . Anderson and Badawi [1] generalized the notion of 2-absorbing ideals to  $n$ -absorbing ideals. A proper ideal  $I$  of  $R$  is called an  *$n$ -absorbing* (resp. *a strongly  $n$ -absorbing*) *ideal* if whenever  $x_1 \cdots x_{n+1} \in I$  for  $x_1, \dots, x_{n+1} \in R$  (resp.  $I_1 \cdots I_{n+1} \subseteq I$  for ideals  $I_1, \dots, I_{n+1}$  of  $R$ ), then there are  $n$  of the  $x_i$ 's (resp.  $n$  of the  $I_i$ 's) whose product is in  $I$ . Badawi et. al. [4] defined a proper ideal  $I$  of  $R$  to be a *2-absorbing primary ideal of  $R$*  if whenever  $a, b, c \in R$  and  $abc \in I$ , then either  $ab \in I$  or  $ac \in \sqrt{I}$  or  $bc \in \sqrt{I}$ . Let  $I$  be a 2-absorbing primary ideal of  $R$ . Then  $P = \sqrt{I}$  is a 2-absorbing ideal of  $R$  by [4, Theorem 2.2]: We say that  $I$  is a  *$P$ -2-absorbing primary ideal of  $R$* . For more studies concerning 2-absorbing (submodules) ideals we refer to [5],[9],[10],[15],[16]. These concepts motivate us to introduce a generalization of uniformly primary ideals. A proper

2010 *Mathematics Subject Classification.* Primary: 13A15; secondary: 13E05; 13F05.

*Key words and phrases.* Uniformly 2-absorbing primary ideal, Noether strongly 2-absorbing primary ideal, 2-absorbing primary ideal.

ideal  $Q$  of  $R$  is said to be a *uniformly 2-absorbing primary ideal* of  $R$  if there exists a positive integer  $n$  such that whenever  $a, b, c \in R$  satisfy  $abc \in Q$ ,  $ab \notin Q$  and  $ac \notin \sqrt{Q}$ , then  $(bc)^n \in Q$ . In particular, if for  $n = 1$  the above property holds, then we say that  $Q$  is a *special 2-absorbing primary ideal* of  $R$ .

In section 2, we introduce the concepts of uniformly 2-absorbing primary ideals and Noether strongly 2-absorbing primary ideals. Then we investigate the relationship between uniformly 2-absorbing primary ideals, Noether strongly 2-absorbing primary ideals and 2-absorbing primary ideals. After that, in Theorem 2.13 we characterize uniformly 2-absorbing primary ideals. We show that if  $Q_1, Q_2$  are uniformly primary ideals of ring  $R$ , then  $Q_1 \cap Q_2$  and  $Q_1 Q_2$  are uniformly 2-absorbing primary ideals of  $R$ , Theorem 2.20. Let  $R = R_1 \times R_2$ , where  $R_1$  and  $R_2$  are rings with  $1 \neq 0$ . It is shown (Theorem 2.31) that a proper ideal  $Q$  of  $R$  is a uniformly 2-absorbing primary ideal of  $R$  if and only if either  $Q = Q_1 \times R_2$  for some uniformly 2-absorbing primary ideal  $Q_1$  of  $R_1$  or  $Q = R_1 \times Q_2$  for some uniformly 2-absorbing primary ideal  $Q_2$  of  $R_2$  or  $Q = Q_1 \times Q_2$  for some uniformly primary ideal  $Q_1$  of  $R_1$  and some uniformly primary ideal  $Q_2$  of  $R_2$ .

In section 3, we give some properties of special 2-absorbing primary ideals. For example, in Theorem 3.5 we show that  $Q$  is a special 2-absorbing primary ideal of  $R$  if and only if for every ideals  $I, J, K$  of  $R$ ,  $IJK \subseteq Q$  implies that either  $IJ \subseteq \sqrt{Q}$  or  $IK \subseteq Q$  or  $JK \subseteq Q$ . We prove that if  $Q$  is a special 2-absorbing primary ideal of  $R$  and  $x \in R \setminus \sqrt{Q}$ , then  $(Q :_R x)$  is a special 2-absorbing primary ideal of  $R$ , Theorem 3.6. It is proved (Theorem 3.7) that an irreducible ideal  $Q$  of  $R$  is special 2-absorbing primary if and only if  $(Q :_R x) = (Q :_R x^2)$  for every  $x \in R \setminus \sqrt{Q}$ . Let  $R$  be a Prüfer domain and  $I$  be an ideal of  $R$ . In Corollary 3.13 we show that  $Q$  is a special 2-absorbing primary ideal of  $R$  if and only if  $Q[X]$  is a special 2-absorbing primary ideal of  $R[X]$ .

## 2. UNIFORMLY 2-ABSORBING PRIMARY IDEALS

Let  $Q$  be a  $P$ -primary ideal of  $R$ . We recall from [6] that  $Q$  is a *Noether strongly primary ideal* of  $R$  if  $P^n \subseteq Q$  for some positive integer  $n$ . We say that  $N$  is the exponent of  $Q$  if  $N$  is the smallest positive integer for which the above property holds and it is denoted by  $\epsilon(Q) = N$ .

**Definition 2.1.** Let  $Q$  be a proper ideal of a commutative ring  $R$ .

- (1)  $Q$  is a *uniformly 2-absorbing primary ideal* of  $R$  if there exists a positive integer  $n$  such that whenever  $a, b, c \in R$  satisfy  $abc \in Q$ ,  $ab \notin Q$  and  $ac \notin \sqrt{Q}$ , then  $(bc)^n \in Q$ . We call that  $N$  is order of  $Q$  if  $N$  is the smallest positive integer for which the above property holds and it is denoted by  $2\text{-ord}_R(Q) = N$  or  $2\text{-ord}(Q) = N$ .
- (2)  $P$ -2-absorbing primary ideal  $Q$  is a *Noether strongly 2-absorbing primary ideal* of  $R$  if  $P^n \subseteq Q$  for some positive integer  $n$ . We say that  $N$  is the exponent of  $Q$  if  $N$  is the smallest positive integer for which the above property holds and it is denoted by  $2\text{-}\epsilon(Q) = N$ .

A *valuation ring* is an integral domain  $V$  such that for every element  $x$  of its field of fractions  $K$ , at least one of  $x$  or  $x^{-1}$  belongs to  $V$ .

**Proposition 2.2.** Let  $V$  be a valuation ring with the quotient field  $K$  and let  $Q$  be a proper ideal of  $V$ .

- (1)  $Q$  is a uniformly 2-absorbing primary ideal of  $V$ ;
- (2) There exists a positive integer  $n$  such that for every  $x, y, z \in K$  whenever  $xyz \in Q$  and  $xy \notin Q$ , then  $xz \in \sqrt{Q}$  or  $(yz)^n \in Q$ .

*Proof.* (1) $\Rightarrow$ (2) Assume that  $Q$  is a uniformly 2-absorbing primary ideal of  $V$ . Let  $xyz \in Q$  for some  $x, y, z \in K$  such that  $xy \notin Q$ . If  $z \notin V$ , then  $z^{-1} \in V$ , since  $V$  is valuation. So  $xyz z^{-1} = xy \in Q$ , a contradiction. Hence  $z \in V$ . If  $x, y \in V$ , then there is nothing to prove. If  $y \notin V$ , then  $xz \in Q \subseteq \sqrt{Q}$ , and if  $x \notin V$ , then  $yz \in Q$ . Consequently we have the claim.

(2) $\Rightarrow$ (1) Is clear.  $\square$

**Proposition 2.3.** Let  $Q_1, Q_2$  be two Noether strongly primary ideals of ring  $R$ . Then  $Q_1 \cap Q_2$  and  $Q_1 Q_2$  are Noether strongly 2-absorbing primary ideals of  $R$  such that  $2\text{-}\mathfrak{e}(Q_1 \cap Q_2) \leq \max\{\mathfrak{e}(Q_1), \mathfrak{e}(Q_2)\}$  and  $2\text{-}\mathfrak{e}(Q_1 Q_2) \leq \mathfrak{e}(Q_1) + \mathfrak{e}(Q_2)$ .

*Proof.* Since  $Q_1, Q_2$  are primary ideals of  $R$ , then  $Q_1 \cap Q_2$  and  $Q_1 Q_2$  are 2-absorbing primary ideals of  $R$ , by [4, Theorem 2.4].  $\square$

**Proposition 2.4.** If  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ , then  $Q$  is a 2-absorbing primary ideal of  $R$ .

*Proof.* Straightforward.  $\square$

**Proposition 2.5.** Let  $R$  be a ring and  $Q$  be a proper ideal of  $R$ .

- (1) If  $Q$  is a 2-absorbing ideal of  $R$ , then
  - (a)  $Q$  is a Noether strongly 2-absorbing primary ideal with  $2\text{-}\mathfrak{e}(Q) \leq 2$ .
  - (b)  $Q$  is a uniformly 2-absorbing primary ideal with  $2\text{-ord}(Q) = 1$ .
- (2) If  $Q$  is a uniformly primary ideal of  $R$ , then it is a uniformly 2-absorbing primary ideal with  $2\text{-ord}(Q) = 1$ .

*Proof.* (1) (a) If  $Q$  is a 2-absorbing ideal, then it is a 2-absorbing primary ideal and  $(\sqrt{Q})^2 \subseteq Q$ , by [3, Theorem 2.4].

(b) Is evident.

(2) Let  $Q$  be a uniformly primary ideal of  $R$  and let  $abc \in Q$  for some  $a, b, c \in R$  such that  $ab \notin Q$  and  $ac \notin \sqrt{Q}$ . Since  $Q$  is primary,  $abc \in Q$  and  $ac \notin \sqrt{Q}$ , then  $b \in Q$ . Therefore  $bc \in Q$ . Consequently  $Q$  is a uniformly 2-absorbing primary ideal with  $2\text{-ord}(Q) = 1$ .  $\square$

**Example 2.6.** Let  $R = K[X, Y]$  where  $K$  is a field. Then  $Q = (X^2, XY, Y^2)R$  is a Noether strongly  $(X, Y)R$ -primary ideal of  $R$  and so it is a Noether strongly 2-absorbing primary ideal of  $R$ .

**Proposition 2.7.** If  $Q$  is a Noether strongly 2-absorbing primary ideal of  $R$ , then  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  and  $2\text{-ord}(Q) \leq 2\text{-}\mathfrak{e}(Q)$ .

*Proof.* Let  $Q$  be a Noether strongly 2-absorbing primary ideal of  $R$ . Now, let  $a, b, c \in R$  such that  $abc \in Q$ ,  $ab \notin Q$ ,  $ac \notin \sqrt{Q}$ . Then  $bc \in \sqrt{Q}$  since  $Q$  is a 2-absorbing primary ideal of  $R$ . Thus  $(bc)^{2\text{-}\mathfrak{e}(Q)} \in (\sqrt{Q})^{2\text{-}\mathfrak{e}(Q)} \subseteq Q$ . Therefore,  $Q$  is a uniformly 2-absorbing primary ideal and also  $2\text{-ord}(Q) \leq 2\text{-}\mathfrak{e}(Q)$ .  $\square$

In the following example, we show that the converse of Proposition 2.7 is not true. We make use of [6, Example 6 and Example 7]

**Example 2.8.** Let  $R$  be a ring of characteristic 2 and  $T = R[X]$  where  $X = \{X_1, X_2, X_3, \dots\}$  is a set of indeterminates over  $R$ . Let  $Q = (\{X_i^2\}_{i=1}^\infty)T$ . By [6, Example 7]  $Q$  is a uniformly  $P$ -primary ideal of  $T$  with  $\text{ord}_T(Q) = 1$  where  $P = (X)T$ . Then  $Q$  is a uniformly 2-absorbing primary ideal of  $T$  with  $2\text{-ord}_T(Q) = 1$ , by Proposition 2.5(2). But  $Q$  is not a Noether strongly 2-absorbing primary ideal since for every positive integer  $n$ ,  $P^n \not\subseteq Q$ .

**Remark 2.9.** Every 2-absorbing ideal of ring  $R$  is a uniformly 2-absorbing primary ideal, but the converse does not necessarily hold. For example, let  $p, q$  be two distinct prime numbers. Then  $p^2q\mathbb{Z}$  is a 2-absorbing primary ideal of  $\mathbb{Z}$ , [4, Corollary 2.12]. On the other hand  $(\sqrt{p^2q\mathbb{Z}})^2 = p^2q^2\mathbb{Z} \subseteq p^2q\mathbb{Z}$ , and so  $p^2q\mathbb{Z}$  is a Noether strongly 2-absorbing primary ideal of  $\mathbb{Z}$ . Hence Proposition 2.7 implies that  $p^2q\mathbb{Z}$  is a uniformly 2-absorbing primary ideal. But, notice that  $p^2q \in p^2q\mathbb{Z}$  and neither  $p^2 \in p^2q\mathbb{Z}$  nor  $pq \in p^2q\mathbb{Z}$  which shows that  $p^2q\mathbb{Z}$  is not a 2-absorbing ideal of  $\mathbb{Z}$ . Also, it is easy to see that  $p^2q\mathbb{Z}$  is not primary and so it is not a uniformly primary ideal of  $\mathbb{Z}$ . Consequently the two concepts of uniformly primary ideals and of uniformly 2-absorbing primary ideals are different in general.

**Proposition 2.10.** *Let  $R$  be a ring and  $Q$  be a proper ideal of  $R$ . If  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ , then one of the following conditions must hold:*

- (1)  $\sqrt{Q} = \mathfrak{p}$  is a prime ideal.
- (2)  $\sqrt{Q} = \mathfrak{p} \cap \mathfrak{q}$ , where  $\mathfrak{p}$  and  $\mathfrak{q}$  are the only distinct prime ideals of  $R$  that are minimal over  $Q$ .

*Proof.* Use [4, Theorem 2.3]. □

Let  $R$  be a ring and  $I$  be an ideal of  $R$ . We denote by  $I^{[n]}$  the ideal of  $R$  generated by the  $n$ -th powers of all elements of  $I$ . If  $n!$  is a unit in  $R$ , then  $I^{[n]} = I^n$ , see [2].

**Theorem 2.11.** *Let  $Q$  be a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  $Q$  is uniformly primary;
- (2) There exists a positive integer  $n$  such that for every ideals  $I, J$  of  $R$ ,  $IJ \subseteq Q$  implies that either  $I \subseteq Q$  or  $J^{[n]} \subseteq Q$ ;
- (3) There exists a positive integer  $n$  such that for every  $a \in R$  either  $a \in Q$  or  $(Q :_R a)^{[n]} \subseteq Q$ ;
- (4) There exists a positive integer  $n$  such that for every  $a \in R$  either  $a^n \in Q$  or  $(Q :_R a) = Q$ .

*Proof.* (1) $\Rightarrow$ (2) Suppose that  $Q$  is uniformly primary with  $\text{ord}(Q) = n$ . Let  $IJ \subseteq Q$  for some ideals  $I, J$  of  $R$ . Assume that neither  $I \subseteq Q$  nor  $J^{[n]} \subseteq Q$ . Then there exist elements  $a \in I \setminus Q$  and  $b^n \in J^{[n]} \setminus Q$ , where  $b \in J$ . Since  $ab \in IJ \subseteq Q$ , then either  $a \in Q$  or  $b^n \in Q$ , which is a contradiction. Therefore either  $I \subseteq Q$  or  $J^{[n]} \subseteq Q$ .

(2) $\Rightarrow$ (3) Note that  $a(Q :_R a) \subseteq Q$  for every  $a \in R$ .

(3) $\Rightarrow$ (1) and (1) $\Leftrightarrow$ (4) have easy verifications. □

**Corollary 2.12.** *Let  $R$  be a ring. Suppose that  $n!$  is a unit in  $R$  for every positive integer  $n$ , and  $Q$  is a proper ideal of  $R$ . The following conditions are equivalent:*

- (1)  $Q$  is uniformly primary;
- (2) There exists a positive integer  $n$  such that for every ideals  $I, J$  of  $R$ ,  $IJ \subseteq Q$  implies that either  $I \subseteq Q$  or  $J^n \subseteq Q$ ;

- (3) There exists a positive integer  $n$  such that for every  $a \in R$  either  $a \in Q$  or  $(Q :_R a)^n \subseteq Q$ ;
- (4) There exists a positive integer  $n$  such that for every  $a \in R$  either  $a^n \in Q$  or  $(Q :_R a) = Q$ .

In the following theorem we characterize uniformly 2-absorbing primary ideals.

**Theorem 2.13.** *Let  $Q$  be a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  $Q$  is uniformly 2-absorbing primary;
- (2) There exists a positive integer  $n$  such that for every  $a, b \in R$  either  $(ab)^n \in Q$  or  $(Q :_R ab) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R b)$ ;
- (3) There exists a positive integer  $n$  such that for every  $a, b \in R$  either  $(ab)^n \in Q$  or  $(Q :_R ab) = (Q :_R a)$  or  $(Q :_R ab) \subseteq (\sqrt{Q} :_R b)$ ;
- (4) There exists a positive integer  $n$  such that for every  $a, b \in R$  and every ideal  $I$  of  $R$ ,  $abI \subseteq Q$  implies that either  $aI \subseteq Q$  or  $bI \subseteq \sqrt{Q}$  or  $(ab)^n \in Q$ ;
- (5) There exists a positive integer  $n$  such that for every  $a, b \in R$  either  $ab \in Q$  or  $(Q :_R ab)^{[n]} \subseteq (\sqrt{Q} :_R a) \cup (Q :_R b^n)$ ;
- (6) There exists a positive integer  $n$  such that for every  $a, b \in R$  either  $ab \in Q$  or  $(Q :_R ab)^{[n]} \subseteq (\sqrt{Q} :_R a)$  or  $(Q :_R ab)^{[n]} \subseteq (Q :_R b^n)$ .

*Proof.* (1) $\Rightarrow$ (2) Suppose that  $Q$  is uniformly 2-absorbing primary with  $2\text{-ord}(Q) = n$ . Assume that  $a, b \in R$  such that  $(ab)^n \notin Q$ . Let  $x \in (Q :_R ab)$ . Thus  $xab \in Q$ , and so either  $xa \in Q$  or  $xb \in \sqrt{Q}$ . Hence  $x \in (Q :_R a)$  or  $x \in (\sqrt{Q} :_R b)$  which shows that  $(Q :_R ab) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R b)$ .

(2) $\Rightarrow$ (3) By the fact that if an ideal is a subset of the union of two ideals, then it is a subset of one of them.

(3) $\Rightarrow$ (4) Suppose that  $n$  is a positive number which exists by part (3). Let  $a, b \in R$  and  $I$  be an ideal of  $R$  such that  $abI \subseteq Q$  and  $(ab)^n \notin Q$ . Then  $I \subseteq (Q :_R ab)$ , and so  $I \subseteq (Q :_R a)$  or  $I \subseteq (\sqrt{Q} :_R b)$ , by (3). Consequently  $aI \subseteq Q$  or  $bI \subseteq \sqrt{Q}$ .

(4) $\Rightarrow$ (1) Is easy.

(1) $\Rightarrow$ (5) Suppose that  $Q$  is uniformly 2-absorbing primary with  $2\text{-ord}(Q) = n$ . Assume that  $a, b \in R$  such that  $ab \notin Q$ . Let  $x \in (Q :_R ab)$ . Then  $abx \in Q$ . So  $ax \in \sqrt{Q}$  or  $(bx)^n \in Q$ . Hence  $x^n \in (\sqrt{Q} :_R a)$  or  $x^n \in (Q :_R b^n)$ . Consequently  $(Q :_R ab)^{[n]} \subseteq (\sqrt{Q} :_R a) \cup (Q :_R b^n)$ ;

(5) $\Rightarrow$ (6) Is similar to the proof of (2) $\Rightarrow$ (3).

(6) $\Rightarrow$ (1) Assume (6). Let  $abc \in Q$  for some  $a, b, c \in R$  such that  $ab \notin Q$ . Then  $c \in (Q :_R ab)$  and thus  $c^n \in (Q :_R ab)^{[n]}$ . So, by part (6) we have that  $c^n \in (\sqrt{Q} :_R a)$  or  $c^n \in (Q :_R b^n)$ . Therefore  $ac \in \sqrt{Q}$  or  $(bc)^n \in Q$ , and so  $Q$  is uniformly 2-absorbing primary.  $\square$

**Corollary 2.14.** *Let  $R$  be a ring. Suppose that  $n!$  is a unit in  $R$  for every positive integer  $n$ , and  $Q$  is a proper ideal of  $R$ . The following conditions are equivalent:*

- (1)  $Q$  is uniformly 2-absorbing primary;
- (2) There exists a positive integer  $n$  such that for every  $a, b \in R$  either  $ab \in Q$  or  $(Q :_R ab)^n \subseteq (\sqrt{Q} :_R a) \cup (Q :_R b^n)$ ;
- (3) There exists a positive integer  $n$  such that for every  $a, b \in R$  either  $ab \in Q$  or  $(Q :_R ab)^n \subseteq (\sqrt{Q} :_R a)$  or  $(Q :_R ab)^n \subseteq (Q :_R b^n)$ .

**Proposition 2.15.** *Let  $Q$  be a uniformly 2-absorbing primary ideal of  $R$  and  $x \in R \setminus Q$  be idempotent. The following conditions hold:*

- (1)  $(\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$ .
- (2)  $(Q :_R x)$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}((Q :_R x)) \leq 2\text{-ord}(Q)$ .

*Proof.* (1) Is easy.

(2) Suppose that  $2\text{-ord}(Q) = n$ . Let  $abc \in (Q :_R x)$  for some  $a, b, c \in R$ . Then  $a(bc)x \in Q$  and so either  $abc \in Q$  or  $ax \in \sqrt{Q}$  or  $(bc)^n x \in Q$ . If  $abc \in Q$ , then either  $ab \in Q \subseteq (Q :_R x)$  or  $ac \in \sqrt{Q} \subseteq \sqrt{(Q :_R x)}$  or  $(bc)^n \in Q \subseteq (Q :_R x)$ . If  $ax \in \sqrt{Q}$ , then  $ac \in (\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$  by part (1). In the third case we have  $(bc)^n \in (Q :_R x)$ . Hence  $(Q :_R x)$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}((Q :_R x)) \leq n$ .  $\square$

**Proposition 2.16.** *Let  $I$  be a proper ideal of ring  $R$ .*

- (1)  $\sqrt{I}$  is a 2-absorbing ideal of  $R$ .
- (2) For every  $a, b, c \in R$ ,  $abc \in I$  implies that  $ab \in \sqrt{I}$  or  $ac \in \sqrt{I}$  or  $bc \in \sqrt{I}$ ;
- (3)  $\sqrt{I}$  is a 2-absorbing primary ideal of  $R$ ;
- (4)  $\sqrt{I}$  is a Noether 2-absorbing primary ideal of  $R$  ( $2\text{-}\mathfrak{c}(\sqrt{I}) = 1$ );
- (5)  $\sqrt{I}$  is a uniformly 2-absorbing primary ideal of  $R$ .

*Proof.* (1) $\Rightarrow$ (2) Is trivial.

(2) $\Rightarrow$ (1) Let  $xyz \in \sqrt{I}$  for some  $x, y, z \in R$ . Then there exists a positive integer  $m$  such that  $x^m y^m z^m \in I$ . So, the hypothesis in (2) implies that  $x^m y^m \in \sqrt{I}$  or  $x^m z^m \in \sqrt{I}$  or  $y^m z^m \in \sqrt{I}$ . Hence  $xy \in \sqrt{I}$  or  $xz \in \sqrt{I}$  or  $yz \in \sqrt{I}$  which shows that  $\sqrt{I}$  is a 2-absorbing ideal.

(1) $\Leftrightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (5) By Proposition 2.7.

(5) $\Rightarrow$ (3) Is easy.  $\square$

**Proposition 2.17.** *If  $Q_1$  is a uniformly  $P$ -primary ideal of  $R$  and  $Q_2$  is a uniformly  $P$ -2-absorbing primary ideal of  $R$  such that  $Q_1 \subseteq Q_2$ , then  $2\text{-ord}(Q_2) \leq \text{ord}(Q_1)$ .*

*Proof.* Let  $\text{ord}(Q_1) = m$  and  $2\text{-ord}(Q_2) = n$ . Then there are  $a, b, c \in R$  such that  $abc \in Q_2$ ,  $ab \notin Q_2$ ,  $ac \notin \sqrt{Q_2}$  and  $(bc)^n \in Q_2$  but  $(bc)^{n-1} \notin Q_2$ . Thus  $bc \in \sqrt{Q_2} = \sqrt{Q_1}$ . Hence  $(bc)^m \in Q_1 \subseteq Q_2$  by [6, Proposition 8]. Therefore,  $n > m - 1$  and so  $n \geq m$ .  $\square$

**Theorem 2.18.** *Let  $R$  be a ring and  $\{Q_i\}_{i \in I}$  be a chain of uniformly  $P$ -2-absorbing primary ideals such that  $\max_{i \in I} \{2\text{-ord}(Q_i)\} = n$ , where  $n$  is a positive integer. Then  $Q = \bigcap_{i \in I} Q_i$  is a uniformly  $P$ -2-absorbing primary ideal of  $R$  with  $2\text{-ord}(Q) \leq n$ .*

*Proof.* It is clear that  $\sqrt{Q} = \bigcap_{i \in I} \sqrt{Q_i} = P$ . Let  $a, b, c \in R$  such that  $abc \in Q$ ,  $ab \notin Q$  and  $(bc)^n \notin Q$ . Since  $\{Q_i\}_{i \in I}$  is a chain, there exists some  $k \in I$  such that  $ab \notin Q_k$  and  $(bc)^n \notin Q_k$ . On the other hand  $Q_k$  is uniformly 2-absorbing primary with  $2\text{-ord}(Q_k) \leq n$ , thus  $ac \in \sqrt{Q_k} = \sqrt{Q}$ , and so  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}(Q) \leq n$ .  $\square$

In the following remark, we show that if  $Q_1$  and  $Q_2$  are uniformly 2-absorbing primary ideals of  $R$ , then  $Q_1 \cap Q_2$  need not be a uniformly 2-absorbing primary ideal of  $R$ .



**Remark 2.19.** Let  $p, q, r$  be distinct prime numbers. Then  $p^2q\mathbb{Z}$  and  $r\mathbb{Z}$  are uniformly 2-absorbing primary ideals of  $\mathbb{Z}$ . Notice that  $p^2qr \in p^2q\mathbb{Z} \cap r\mathbb{Z}$  and neither  $p^2q \in p^2q\mathbb{Z} \cap r\mathbb{Z}$  nor  $p^2r \in \sqrt{p^2q\mathbb{Z} \cap r\mathbb{Z}} = p\mathbb{Z} \cap q\mathbb{Z} \cap r\mathbb{Z}$  nor  $qr \in \sqrt{p^2q\mathbb{Z} \cap r\mathbb{Z}} = p\mathbb{Z} \cap q\mathbb{Z} \cap r\mathbb{Z}$ . Hence  $p^2q\mathbb{Z} \cap r\mathbb{Z}$  is not a 2-absorbing primary ideal of  $\mathbb{Z}$  which shows that it is not a uniformly 2-absorbing primary ideal of  $\mathbb{Z}$ .

**Theorem 2.20.** Let  $Q_1, Q_2$  be uniformly primary ideals of ring  $R$ .

- (1)  $Q_1 \cap Q_2$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}(Q_1 \cap Q_2) \leq \max\{\text{ord}(Q_1), \text{ord}(Q_2)\}$ .
- (2)  $Q_1 Q_2$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}(Q_1 Q_2) \leq \text{ord}(Q_1) + \text{ord}(Q_2)$ .

*Proof.* (1) Let  $Q_1, Q_2$  be uniformly primary. Set  $n = \max\{\text{ord}(Q_1), \text{ord}(Q_2)\}$ . Assume that for some  $a, b, c \in R$ ,  $abc \in Q_1 \cap Q_2$ ,  $ab \notin Q_1 \cap Q_2$  and  $ac \notin \sqrt{Q_1 \cap Q_2}$ . Since  $Q_1$  and  $Q_2$  are primary ideals of  $R$ , then  $Q_1 \cap Q_2$  is 2-absorbing primary by [4, Theorem 2.4]. Therefore  $bc \in \sqrt{Q_1 \cap Q_2} = \sqrt{Q_1} \cap \sqrt{Q_2}$ . By [6, Proposition 8] we have that  $(bc)^{\text{ord}(Q_1)} \in Q_1$  and  $(bc)^{\text{ord}(Q_2)} \in Q_2$ . Hence  $(bc)^n \in Q_1 \cap Q_2$  which shows that  $Q_1 \cap Q_2$  is uniformly 2-absorbing primary and  $2\text{-ord}(Q_1 \cap Q_2) \leq n$ .

(2) Similar to the proof in (1).  $\square$

We recall from [7], if  $R$  is an integral domain and  $P$  is a prime ideal of  $R$  that can be generated by a regular sequence of  $R$ . Then, for each positive integer  $n$ , the ideal  $P^n$  is a  $P$ -primary ideal of  $R$ .

**Lemma 2.21.** ([6, Corollary 4]) Let  $R$  be a ring and  $P$  be a prime ideal of  $R$ . If  $P^n$  is a  $P$ -primary ideal of  $R$  for some positive integer  $n$ , then  $P^n$  is a uniformly primary ideal of  $R$  with  $\text{ord}(P^n) \leq n$ .

**Corollary 2.22.** Let  $R$  be a ring and  $P_1, P_2$  be prime ideals of  $R$ . If  $P_1^n$  is a  $P_1$ -primary ideal of  $R$  for some positive integer  $n$  and  $P_2^m$  is a  $P_2$ -primary ideal of  $R$  for some positive integer  $m$ , then  $P_1^n P_2^m$  and  $P_1^n \cap P_2^m$  are uniformly 2-absorbing primary ideals of  $R$  with  $2\text{-ord}(P_1^n P_2^m) \leq n+m$  and  $2\text{-ord}(P_1^n \cap P_2^m) \leq \max\{n, m\}$ .

*Proof.* By Theorem 2.20 and Lemma 2.21.  $\square$

**Proposition 2.23.** Let  $f : R \rightarrow R'$  be a homomorphism of commutative rings. Then the following statements hold:

- (1) If  $Q'$  is a uniformly 2-absorbing primary ideal of  $R'$ , then  $f^{-1}(Q')$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}_R(f^{-1}(Q')) \leq 2\text{-ord}_{R'}(Q')$ .
- (2) If  $f$  is an epimorphism and  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  containing  $\ker(f)$ , then  $f(Q)$  is a uniformly 2-absorbing primary ideal of  $R'$  with  $2\text{-ord}_{R'}(f(Q)) \leq 2\text{-ord}_R(Q)$ .

*Proof.* (1) Set  $N = 2\text{-ord}_{R'}(Q')$ . Let  $a, b, c \in R$  such that  $abc \in f^{-1}(Q')$ ,  $ab \notin f^{-1}(Q')$  and  $ac \notin \sqrt{f^{-1}(Q')} = f^{-1}(\sqrt{Q'})$ . Then  $f(abc) = f(a)f(b)f(c) \in Q'$ ,  $f(ab) = f(a)f(b) \notin Q'$  and  $f(ac) = f(a)f(c) \notin \sqrt{Q'}$ . Since  $Q'$  is a uniformly 2-absorbing primary ideal of  $R'$ , then  $f^N(bc) \in Q'$ . Then  $f((bc)^N) \in Q'$  and so  $(bc)^N \in f^{-1}(Q')$ . Thus  $f^{-1}(Q')$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}_R(f^{-1}(Q')) \leq N = 2\text{-ord}_{R'}(Q')$ .

(2) Set  $N = 2\text{-ord}_R(Q)$ . Let  $a, b, c \in R'$  such that  $abc \in f(Q)$ ,  $ab \notin f(Q)$  and  $ac \notin \sqrt{f(Q)}$ . Since  $f$  is an epimorphism, then there exist  $x, y, z \in R$  such that  $f(x) = a$ ,  $f(y) = b$  and  $f(z) = c$ . Then  $f(xyz) = abc \in f(Q)$ ,  $f(xy) = ab \notin f(Q)$

and  $f(xz) = ac \notin \sqrt{f(Q)}$ . Since  $\ker(f) \subseteq Q$ , then  $xyz \in Q$ . Also  $xy \notin Q$ , and  $xz \notin \sqrt{Q}$ , since  $f(\sqrt{Q}) \subseteq \sqrt{f(Q)}$ . Then  $(yz)^N \in Q$  since  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ . Thus  $f((yz)^N) = (f(y)f(z))^N = (bc)^N \in f(Q)$ . Therefore,  $f(Q)$  is a uniformly 2-absorbing primary ideal of  $R'$ . Moreover  $2\text{-ord}_{R'}(f(Q)) \leq N = 2\text{-ord}_R(Q)$ .  $\square$

As an immediate consequence of Proposition 2.23 we have the following result:

**Corollary 2.24.** *Let  $R$  be a ring and  $Q$  be an ideal of  $R$ .*

- (1) *If  $R'$  is a subring of  $R$  and  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ , then  $Q \cap R'$  is a uniformly 2-absorbing primary ideal of  $R'$  with  $2\text{-ord}_{R'}(Q \cap R') \leq 2\text{-ord}_R(Q)$ .*
- (2) *Let  $I$  be an ideal of  $R$  with  $I \subseteq Q$ . Then  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  if and only if  $Q/I$  is a uniformly 2-absorbing primary ideal of  $R/I$ .*

**Corollary 2.25.** *Let  $Q$  be an ideal of ring  $R$ . Then  $\langle Q, X \rangle$  is a uniformly 2-absorbing primary ideal of  $R[X]$  if and only if  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ .*

*Proof.* By Corollary 2.24(2) and regarding the isomorphism  $\langle Q, X \rangle / \langle X \rangle \simeq Q$  in  $R[X] / \langle X \rangle \simeq R$  we have the result.  $\square$

**Corollary 2.26.** *Let  $R$  be a ring,  $Q$  a proper ideal of  $R$  and  $X = \{X_i\}_{i \in I}$  a collection of indeterminates over  $R$ . If  $QR[X]$  is a uniformly 2-absorbing primary ideal of  $R[X]$ , then  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}_R(Q) \leq 2\text{-ord}_{R[X]}(QR[X])$ .*

*Proof.* It is clear from Corollary 2.24(1).  $\square$

**Proposition 2.27.** *Let  $S$  be a multiplicatively closed subset of  $R$  and  $Q$  be a proper ideal of  $R$ . Then the following conditions hold:*

- (1) *If  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  such that  $Q \cap S = \emptyset$ , then  $S^{-1}Q$  is a uniformly 2-absorbing primary ideal of  $S^{-1}R$  with  $2\text{-ord}(S^{-1}Q) \leq 2\text{-ord}(Q)$ .*
- (2) *If  $S^{-1}Q$  is a uniformly 2-absorbing primary ideal of  $S^{-1}R$  and  $S \cap Z_Q(R) = \emptyset$ , then  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}(Q) \leq 2\text{-ord}(S^{-1}Q)$ .*

*Proof.* (1) Set  $N := 2\text{-ord}(Q)$ . Let  $a, b, c \in R$  and  $s, t, k \in S$  such that  $\frac{a}{s} \frac{b}{t} \frac{c}{k} \in S^{-1}Q$ ,  $\frac{a}{s} \frac{b}{t} \notin S^{-1}Q$ ,  $\frac{a}{s} \frac{c}{k} \notin \sqrt{S^{-1}Q} = S^{-1}\sqrt{Q}$ . Thus there is  $u \in S$  such that  $uabc \in Q$ . By assumptions we have that  $uab \notin Q$  and  $uac \notin \sqrt{Q}$ . Since  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ , then  $(bc)^N \in Q$ . Hence  $(\frac{b}{t} \frac{c}{k})^N \in S^{-1}Q$ . Consequently,  $S^{-1}Q$  is a uniformly 2-absorbing primary ideal of  $S^{-1}R$  and  $2\text{-ord}(S^{-1}Q) \leq N = 2\text{-ord}(Q)$ .

(2) Set  $N := 2\text{-ord}(S^{-1}Q)$ . Let  $a, b, c \in R$  such that  $abc \in Q$ ,  $ab \notin Q$  and  $ac \notin \sqrt{Q}$ . Then  $\frac{abc}{1} = \frac{a}{1} \frac{b}{1} \frac{c}{1} \in S^{-1}Q$ ,  $\frac{ab}{1} = \frac{a}{1} \frac{b}{1} \notin S^{-1}Q$  and  $\frac{ac}{1} = \frac{a}{1} \frac{c}{1} \notin \sqrt{S^{-1}Q} = S^{-1}\sqrt{Q}$ , because  $S \cap Z_Q(R) = \emptyset$  and  $S \cap Z_{\sqrt{Q}}(R) = \emptyset$ . Since  $S^{-1}Q$  is a uniformly 2-absorbing primary ideal of  $S^{-1}R$ , then  $(\frac{b}{1} \frac{c}{1})^N = \frac{(bc)^N}{1} \in S^{-1}Q$ . Then there exists  $u \in S$  such that  $u(bc)^N \in Q$ . Hence  $(bc)^N \in Q$  because  $S \cap Z_Q(R) = \emptyset$ . Thus  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  and  $2\text{-ord}(Q) \leq N = 2\text{-ord}(S^{-1}Q)$ .  $\square$



**Proposition 2.28.** *Let  $Q$  be a 2-absorbing primary ideal of ring  $R$  and  $P = \sqrt{Q}$  be a finitely generated ideal of  $R$ . Then  $Q$  is a Noether strongly 2-absorbing primary ideal of  $R$ . Thus  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ .*

*Proof.* It is clear from [14, Lemma 8.21] and Proposition 2.7.  $\square$

**Corollary 2.29.** *Let  $R$  be a Noetherian ring and  $Q$  a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ ;
- (2)  $Q$  is a Noether strongly 2-absorbing primary ideal of  $R$ ;
- (3)  $Q$  is a 2-absorbing primary ideal of  $R$ .

*Proof.* Apply Proposition 2.7 and Proposition 2.28.  $\square$

We recall from [8] the construction of idealization of a module. Let  $R$  be a ring and  $M$  be an  $R$ -module. Then  $R(+)M = R \times M$  is a ring with identity  $(1, 0)$  under addition defined by  $(r, m) + (s, n) = (r + s, m + n)$  and multiplication defined by  $(r, m)(s, n) = (rs, rn + sm)$ . Note that  $\sqrt{I}(+)M = \sqrt{I(+)M}$ .

**Proposition 2.30.** *Let  $R$  be a ring,  $Q$  be a proper ideal of  $R$  and  $M$  be an  $R$ -module. The following conditions are equivalent:*

- (1)  $Q(+)M$  is a uniformly 2-absorbing primary ideal of  $R(+)M$ ;
- (2)  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ .

*Proof.* The proof is routine.  $\square$

**Theorem 2.31.** *Let  $R = R_1 \times R_2$ , where  $R_1$  and  $R_2$  are rings with  $1 \neq 0$ . Let  $Q$  be a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ ;
- (2) Either  $Q = Q_1 \times R_2$  for some uniformly 2-absorbing primary ideal  $Q_1$  of  $R_1$  or  $Q = R_1 \times Q_2$  for some uniformly 2-absorbing primary ideal  $Q_2$  of  $R_2$  or  $Q = Q_1 \times Q_2$  for some uniformly primary ideal  $Q_1$  of  $R_1$  and some uniformly primary ideal  $Q_2$  of  $R_2$ .

*Proof.* (1) $\Rightarrow$ (2) Assume that  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}_R(Q) = n$ . We know that  $Q$  is in the form of  $Q_1 \times Q_2$  for some ideal  $Q_1$  of  $R_1$  and some ideal  $Q_2$  of  $R_2$ . Suppose that  $Q_2 = R_2$ . Since  $Q$  is a proper ideal of  $R$ ,  $Q_1 \neq R_1$ . Let  $R' = \frac{R}{\{0\} \times R_2}$ . Then  $Q' = \frac{Q}{\{0\} \times R_2}$  is a uniformly 2-absorbing primary ideal of  $R'$  by Corollary 2.24(2). Since  $R'$  is ring-isomorphic to  $R_1$  and  $Q_1 \simeq Q'$ ,  $Q_1$  is a uniformly 2-absorbing primary ideal of  $R_1$ . Suppose that  $Q_1 = R_1$ . Since  $Q$  is a proper ideal of  $R$ ,  $Q_2 \neq R_2$ . By a similar argument as in the previous case,  $Q_2$  is a uniformly 2-absorbing primary ideal of  $R_2$ . Hence assume that  $Q_1 \neq R_1$  and  $Q_2 \neq R_2$ . We claim that  $Q_1$  is a uniformly primary ideal of  $R_1$ . Assume that  $x, y \in R_1$  such that  $xy \in Q_1$  but  $x \notin Q_1$ . Notice that  $(x, 1)(1, 0)(y, 1) = (xy, 0) \in Q$ , but neither  $(x, 1)(1, 0) = (x, 0) \in Q$  nor  $(x, 1)(y, 1) = (xy, 1) \in \sqrt{Q}$ . So  $[(1, 0)(y, 1)]^n = (y^n, 0) \in Q$ . Therefore  $y^n \in Q_1$ . Thus  $Q_1$  is a uniformly primary ideal of  $R_1$  with  $\text{ord}_{R_1}(Q_1) \leq n$ . Now, we claim that  $Q_2$  is a uniformly primary ideal of  $R_2$ . Suppose that for some  $z, w \in R_2$ ,  $zw \in Q_2$  but  $z \notin Q_2$ . Notice that  $(1, z)(0, 1)(1, w) = (0, zw) \in Q$ , but neither  $(1, z)(0, 1) = (0, z) \in Q$  nor  $(1, z)(1, w) = (1, zw) \in \sqrt{Q}$ . Therefore  $[(0, 1)(1, w)]^n = (0, w^n) \in Q$ , and so  $w^n \in Q_2$  which shows that  $Q_2$  is a uniformly primary ideal of  $R_2$  with  $\text{ord}_{R_2}(Q_2) \leq n$ . Consequently when  $Q_1 \neq R_1$  and  $Q_2 \neq R_2$  we have that  $\max\{\text{ord}_{R_1}(Q_1), \text{ord}_{R_2}(Q_2)\} \leq 2\text{-ord}_R(Q)$ .

(2) $\Rightarrow$ (1) If  $Q = Q_1 \times R_2$  for some uniformly 2-absorbing primary ideal  $Q_1$  of  $R_1$ , or  $Q = R_1 \times Q_2$  for some uniformly 2-absorbing primary ideal  $Q_2$  of  $R_2$ , then it is clear that  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ . Hence assume that  $Q = Q_1 \times Q_2$  for some uniformly primary ideal  $Q_1$  of  $R_1$  and some uniformly primary ideal  $Q_2$  of  $R_2$ . Then  $Q'_1 = Q_1 \times R_2$  and  $Q'_2 = R_1 \times Q_2$  are uniformly primary ideals of  $R$  with  $\text{ord}_R(Q'_1) \leq \text{ord}_{R_1}(Q_1)$  and  $\text{ord}_R(Q'_2) \leq \text{ord}_{R_2}(Q_2)$ . Hence  $Q'_1 \cap Q'_2 = Q_1 \times Q_2 = Q$  is a uniformly 2-absorbing primary ideal of  $R$  with  $2\text{-ord}_R(Q) \leq \max\{\text{ord}_{R_1}(Q_1), \text{ord}_{R_2}(Q_2)\}$  by Theorem 2.20.  $\square$

**Lemma 2.32.** *Let  $R = R_1 \times R_2 \times \cdots \times R_n$ , where  $R_1, R_2, \dots, R_n$  are rings with  $1 \neq 0$ . A proper ideal  $Q$  of  $R$  is a uniformly primary ideal of  $R$  if and only if  $Q = \times_{i=1}^n Q_i$  such that for some  $k \in \{1, 2, \dots, n\}$ ,  $Q_k$  is a uniformly primary ideal of  $R_k$ , and  $Q_i = R_i$  for every  $i \in \{1, 2, \dots, n\} \setminus \{k\}$ .*

*Proof.* ( $\Rightarrow$ ) Let  $Q$  be a uniformly primary ideal of  $R$  with  $\text{ord}_R(Q) = m$ . We know  $Q = \times_{i=1}^n Q_i$  where for every  $1 \leq i \leq n$ ,  $Q_i$  is an ideal of  $R_i$ , respectively. Assume that  $Q_r$  is a proper ideal of  $R_r$  and  $Q_s$  is a proper ideal of  $R_s$  for some  $1 \leq r < s \leq n$ . Since

$$(0, \dots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \dots, 0) = (0, \dots, 0) \in Q,$$

then either  $(0, \dots, 0, \overbrace{1_{R_r}}^{r\text{-th}}, 0, \dots, 0) \in Q$  or  $(0, \dots, 0, \overbrace{1_{R_s}}^{s\text{-th}}, 0, \dots, 0)^m \in Q$ , which is a contradiction. Hence exactly one of the  $Q_i$ 's is proper, say  $Q_k$ . Now, we show that  $Q_k$  is a uniformly primary ideal of  $R_k$ . Let  $ab \in Q_k$  for some  $a, b \in R_k$  such that  $a \notin Q_k$ . Therefore

$$(0, \dots, 0, \overbrace{a}^{k\text{-th}}, 0, \dots, 0)(0, \dots, 0, \overbrace{b}^{k\text{-th}}, 0, \dots, 0) = (0, \dots, 0, \overbrace{ab}^{k\text{-th}}, 0, \dots, 0) \in Q,$$

but  $(0, \dots, 0, \overbrace{a}^{k\text{-th}}, 0, \dots, 0) \notin Q$ , and so  $(0, \dots, 0, \overbrace{b}^{k\text{-th}}, 0, \dots, 0)^m \in Q$ . Thus  $b^m \in Q_k$  which implies that  $Q_k$  is a uniformly primary ideals of  $R_k$  with  $\text{ord}_{R_k}(Q_k) \leq m$ .

( $\Leftarrow$ ) Is easy.  $\square$

**Theorem 2.33.** *Let  $R = R_1 \times R_2 \times \cdots \times R_n$ , where  $2 \leq n < \infty$ , and  $R_1, R_2, \dots, R_n$  are rings with  $1 \neq 0$ . For a proper ideal  $Q$  of  $R$  the following conditions are equivalent:*

- (1)  $Q$  is a uniformly 2-absorbing primary ideal of  $R$ .
- (2) Either  $Q = \times_{t=1}^n Q_t$  such that for some  $k \in \{1, 2, \dots, n\}$ ,  $Q_k$  is a uniformly 2-absorbing primary ideal of  $R_k$ , and  $Q_t = R_t$  for every  $t \in \{1, 2, \dots, n\} \setminus \{k\}$  or  $Q = \times_{t=1}^n Q_t$  such that for some  $k, m \in \{1, 2, \dots, n\}$ ,  $Q_k$  is a uniformly primary ideal of  $R_k$ ,  $Q_m$  is a uniformly primary ideal of  $R_m$ , and  $Q_t = R_t$  for every  $t \in \{1, 2, \dots, n\} \setminus \{k, m\}$ .

*Proof.* We use induction on  $n$ . For  $n = 2$  the result holds by Theorem 2.31. Then let  $3 \leq n < \infty$  and suppose that the result is valid when  $K = R_1 \times \cdots \times R_{n-1}$ . We show that the result holds when  $R = K \times R_n$ . By Theorem 2.31,  $Q$  is a uniformly 2-absorbing primary ideal of  $R$  if and only if either  $Q = L \times R_n$  for some uniformly 2-absorbing primary ideal  $L$  of  $K$  or  $Q = K \times L_n$  for some uniformly 2-absorbing

primary ideal  $L_n$  of  $R_n$  or  $Q = L \times L_n$  for some uniformly primary ideal  $L$  of  $K$  and some uniformly primary ideal  $L_n$  of  $R_n$ . Notice that by Lemma 2.32, a proper ideal  $L$  of  $K$  is a uniformly primary ideal of  $K$  if and only if  $L = \times_{t=1}^{n-1} Q_t$  such that for some  $k \in \{1, 2, \dots, n-1\}$ ,  $Q_k$  is a uniformly primary ideal of  $R_k$ , and  $Q_t = R_t$  for every  $t \in \{1, 2, \dots, n-1\} \setminus \{k\}$ . Consequently we reach the claim.  $\square$

### 3. SPECIAL 2-ABSORBING PRIMARY IDEALS

**Definition 3.1.** We say that a proper ideal  $Q$  of a ring  $R$  is *special 2-absorbing primary* if it is uniformly 2-absorbing primary with  $2\text{-ord}(Q) = 1$ .

**Remark 3.2.** By Proposition 2.5(2), every primary ideal is a special 2-absorbing primary ideal. But the converse is not true in general. For example, let  $p, q$  be two distinct prime numbers. Then  $pq\mathbb{Z}$  is a 2-absorbing ideal of  $\mathbb{Z}$  and so it is a special 2-absorbing primary ideal of  $\mathbb{Z}$ , by Proposition 2.5(1). Clearly  $pq\mathbb{Z}$  is not primary.

Recall that a prime ideal  $\mathfrak{p}$  of  $R$  is called *divided prime* if  $\mathfrak{p} \subset xR$  for every  $x \in R \setminus \mathfrak{p}$ .

**Proposition 3.3.** Let  $Q$  be a special 2-absorbing primary ideal of  $R$  such that  $\sqrt{Q} = \mathfrak{p}$  is a divided prime ideal of  $R$ . Then  $Q$  is a  $\mathfrak{p}$ -primary ideal of  $R$ .

*Proof.* Let  $xy \in Q$  for some  $x, y \in R$  such that  $y \notin \mathfrak{p}$ . Then  $x \in \mathfrak{p}$ . Since  $\mathfrak{p}$  is a divided prime ideal,  $\mathfrak{p} \subset yR$  and so there exists  $r \in R$  such that  $x = ry$ . Hence  $xy = ry^2 \in Q$ . Since  $Q$  is special 2-absorbing primary and  $y \notin \mathfrak{p}$ , then  $x = ry \in Q$ . Consequently  $Q$  is a  $\mathfrak{p}$ -primary ideal of  $R$ .  $\square$

**Remark 3.4.** Let  $p, q$  be distinct prime numbers. Then by [4, Theorem 2.4] we can deduce that  $p\mathbb{Z} \cap q^2\mathbb{Z}$  is a 2-absorbing primary ideal of  $\mathbb{Z}$ . Since  $pq^2 \in p\mathbb{Z} \cap q^2\mathbb{Z}$ ,  $pq \notin p\mathbb{Z} \cap q^2\mathbb{Z}$  and  $q^2 \notin p\mathbb{Z} \cap q^2\mathbb{Z}$ , then  $p\mathbb{Z} \cap q^2\mathbb{Z}$  is not a special 2-absorbing primary ideal of  $\mathbb{Z}$ .

Notice that for  $n = 1$  we have that  $I^{[n]} = I$ .

**Theorem 3.5.** Let  $Q$  be a proper ideal of  $R$ . Then the following conditions are equivalent:

- (1)  $Q$  is special 2-absorbing primary;
- (2) For every  $a, b \in R$  either  $ab \in Q$  or  $(Q :_R ab) = (Q :_R a)$  or  $(Q :_R ab) \subseteq (\sqrt{Q} :_R b)$ ;
- (3) For every  $a, b \in R$  and every ideal  $I$  of  $R$ ,  $abI \subseteq Q$  implies that either  $ab \in Q$  or  $aI \subseteq Q$  or  $bI \subseteq \sqrt{Q}$ ;
- (4) For every  $a \in R$  and every ideal  $I$  of  $R$  either  $aI \subseteq Q$  or  $(Q :_R aI) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R I)$ ;
- (5) For every  $a \in R$  and every ideal  $I$  of  $R$  either  $aI \subseteq Q$  or  $(Q :_R aI) = (Q :_R a)$  or  $(Q :_R aI) \subseteq (\sqrt{Q} :_R I)$ ;
- (6) For every  $a \in R$  and every ideals  $I, J$  of  $R$ ,  $aIJ \subseteq Q$  implies that either  $aI \subseteq Q$  or  $IJ \subseteq \sqrt{Q}$  or  $aJ \subseteq Q$ ;
- (7) For every ideals  $I, J$  of  $R$  either  $IJ \subseteq \sqrt{Q}$  or  $(Q :_R IJ) \subseteq (Q :_R I) \cup (Q :_R J)$ ;
- (8) For every ideals  $I, J$  of  $R$  either  $IJ \subseteq \sqrt{Q}$  or  $(Q :_R IJ) = (Q :_R I)$  or  $(Q :_R IJ) = (Q :_R J)$ ;
- (9) For every ideals  $I, J, K$  of  $R$ ,  $IJK \subseteq Q$  implies that either  $IJ \subseteq \sqrt{Q}$  or  $IK \subseteq Q$  or  $JK \subseteq Q$ .

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By Theorem 2.13.

(3) $\Rightarrow$ (4) Let  $a \in R$  and  $I$  be an ideal of  $R$  such that  $aI \not\subseteq Q$ . Suppose that  $x \in (Q :_R aI)$ . Then  $axI \subseteq Q$ , and so by part (3) we have that  $x \in (Q :_R a)$  or  $x \in (\sqrt{Q} :_R I)$ . Therefore  $(Q :_R aI) \subseteq (Q :_R a) \cup (\sqrt{Q} :_R I)$ .

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$ (1) Have straightforward proofs.  $\square$

**Theorem 3.6.** *Let  $Q$  be a special 2-absorbing primary ideal of  $R$  and  $x \in R \setminus \sqrt{Q}$ . The following conditions hold:*

- (1)  $(Q :_R x) = (Q :_R x^n)$  for every  $n \geq 2$ .
- (2)  $(\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$ .
- (3)  $(Q :_R x)$  is a special 2-absorbing primary ideal of  $R$ .

*Proof.* (1) Clearly  $(Q :_R x) \subseteq (Q :_R x^n)$  for every  $n \geq 2$ . For the converse inclusion we use induction on  $n$ . First we get  $n = 2$ . Let  $r \in (Q :_R x^2)$ . Then  $rx^2 \in Q$ , and so either  $rx \in Q$  or  $x^2 \in \sqrt{Q}$ . Notice that  $x^2 \in \sqrt{Q}$  implies that  $x \in \sqrt{Q}$  which is a contradiction. Therefore  $rx \in Q$  and so  $r \in (Q :_R x)$ . Therefore  $(Q :_R x) = (Q :_R x^2)$ . Now, assume  $n > 2$  and suppose that the claim holds for  $n - 1$ , i.e.  $(Q :_R x) = (Q :_R x^{n-1})$ . Let  $r \in (Q :_R x^n)$ . Then  $rx^n \in Q$ . Since  $x \notin \sqrt{Q}$ , then we have either  $rx^{n-1} \in Q$  or  $rx \in Q$ . Both two cases implies that  $r \in (Q :_R x)$ . Consequently  $(Q :_R x) = (Q :_R x^n)$ .

(2) It is easy to investigate that  $\sqrt{(Q :_R x)} \subseteq (\sqrt{Q} :_R x)$ . Let  $r \in (\sqrt{Q} :_R x)$ . Then there exists a positive integer  $m$  such that  $(rx)^m \in Q$ . So, by part (1) we have that  $r^m \in (Q :_R x)$ . Hence  $r \in \sqrt{(Q :_R x)}$ . Thus  $(\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$ .

(3) Let  $abc \in (Q :_R x)$  for some  $a, b, c \in R$ . Then  $ax(bc) \in Q$  and so  $ax \in Q$  or  $abc \in Q$  or  $bcx \in \sqrt{Q}$ . In the first case we have  $ab \in (Q :_R x)$ . If  $abc \in Q$ , then either  $ab \in Q \subseteq (Q :_R x)$  or  $ac \in Q \subseteq (Q :_R x)$  or  $bc \in \sqrt{Q} \subseteq \sqrt{(Q :_R x)}$ . In the third case we have  $bc \in (\sqrt{Q} :_R x) = \sqrt{(Q :_R x)}$  by part (2). Therefore  $(Q :_R x)$  is a special 2-absorbing primary ideal of  $R$ .  $\square$

**Theorem 3.7.** *Let  $Q$  be an irreducible ideal of  $R$ . Then  $Q$  is special 2-absorbing primary if and only if  $(Q :_R x) = (Q :_R x^2)$  for every  $x \in R \setminus \sqrt{Q}$ .*

*Proof.* ( $\Rightarrow$ ) By Theorem 3.6.

( $\Leftarrow$ ) Let  $abc \in Q$  for some  $a, b, c \in R$  such that neither  $ab \in Q$  nor  $ac \in Q$  nor  $bc \in \sqrt{Q}$ . We search for a contradiction. Since  $bc \notin \sqrt{Q}$ , then  $b \notin \sqrt{Q}$ . So, by our hypothesis we have  $(Q :_R b) = (Q :_R b^2)$ . Let  $r \in (Q + Rab) \cap (Q + Rac)$ . Then there are  $q_1, q_2 \in Q$  and  $r_1, r_2 \in R$  such that  $r = q_1 + r_1ab = q_2 + r_2ac$ . Hence  $q_1b + r_1ab^2 = q_2b + r_2abc \in Q$ . Thus  $r_1ab^2 \in Q$ , i.e.,  $r_1a \in (Q :_R b^2) = (Q :_R b)$ . Therefore  $r_1ab \in Q$  and so  $r = q_1 + r_1ab \in Q$ . Then  $Q = (Q + Rab) \cap (Q + Rac)$ , which contradicts the assumption that  $Q$  is irreducible.  $\square$

A ring  $R$  is said to be a *Boolean ring* if  $x = x^2$  for all  $x \in R$ . It is famous that every prime ideal in a Boolean ring  $R$  is maximal. Notice that every ideal of a Boolean ring  $R$  is radical. So, every (uniformly) 2-absorbing primary ideal of  $R$  is a 2-absorbing ideal of  $R$ .

**Corollary 3.8.** *Let  $R$  be a Boolean ring. Then every irreducible ideal of  $R$  is a maximal ideal.*

*Proof.* Let  $I$  be an irreducible ideal of  $R$ . Thus, Theorem 3.7 implies that  $I$  is special 2-absorbing primary. Therefore by Proposition 2.10, either  $I = \sqrt{I}$  is a

maximal ideal or is the intersection of two distinct maximal ideals. Since  $I$  is irreducible, then  $I$  cannot be in the second form. Hence  $I$  is a maximal ideal.  $\square$

*Proof.* By Proposition 2.10 and Theorem 3.7.  $\square$

**Proposition 3.9.** *Let  $Q$  be a special 2-absorbing primary ideal of  $R$  and  $\mathfrak{p}, \mathfrak{q}$  be distinct prime ideals of  $R$ .*

- (1) *If  $\sqrt{Q} = \mathfrak{p}$ , then  $\{(Q :_R x) \mid x \in R \setminus \mathfrak{p}\}$  is a totally ordered set.*
- (2) *If  $\sqrt{Q} = \mathfrak{p} \cap \mathfrak{q}$ , then  $\{(Q :_R x) \mid x \in R \setminus \mathfrak{p} \cup \mathfrak{q}\}$  is a totally ordered set.*

*Proof.* (1) Let  $x, y \in R \setminus \mathfrak{p}$ . Then  $xy \in R \setminus \mathfrak{p}$ . It is clear that  $(Q :_R x) \cup (Q :_R y) \subseteq (Q :_R xy)$ . Assume that  $r \in (Q :_R xy)$ . Therefore  $rx, ry \in Q$ , whence  $rx \in Q$  or  $ry \in Q$ , because  $xy \notin \sqrt{Q}$ . Consequently  $(Q :_R xy) = (Q :_R x) \cup (Q :_R y)$ . Thus, either  $(Q :_R xy) = (Q :_R x)$  or  $(Q :_R xy) = (Q :_R y)$ , and so either  $(Q :_R y) \subseteq (Q :_R x)$  or  $(Q :_R x) \subseteq (Q :_R y)$ .

(2) Is similar to the proof of (1).  $\square$

**Corollary 3.10.** *Let  $f : R \rightarrow R'$  be a homomorphism of commutative rings. Then the following statements hold:*

- (1) *If  $Q'$  is a special 2-absorbing primary ideal of  $R'$ , then  $f^{-1}(Q')$  is a special 2-absorbing primary ideal of  $R$ .*
- (2) *If  $f$  is an epimorphism and  $Q$  is a special 2-absorbing primary ideal of  $R$  containing  $\ker(f)$ , then  $f(Q)$  is a special 2-absorbing primary ideal of  $R'$ .*

*Proof.* By Proposition 2.23.  $\square$

Let  $R$  be a ring with identity. We recall that if  $f = a_0 + a_1X + \cdots + a_tX^t$  is a polynomial on the ring  $R$ , then *content* of  $f$  is defined as the ideal of  $R$ , generated by the coefficients of  $f$ , i.e.  $c(f) = (a_0, a_1, \dots, a_t)$ . Let  $T$  be an  $R$ -algebra and  $c$  the function from  $T$  to the ideals of  $R$  defined by  $c(f) = \cap\{I \mid I \text{ is an ideal of } R \text{ and } f \in IT\}$  known as the content of  $f$ . Note that the content function  $c$  is nothing but the generalization of the content of a polynomial  $f \in R[X]$ . The  $R$ -algebra  $T$  is called a *content  $R$ -algebra* if the following conditions hold:

- (1) For all  $f \in T$ ,  $f \in c(f)T$ .
- (2) (Faithful flatness) For any  $r \in R$  and  $f \in T$ , the equation  $c(rf) = rc(f)$  holds and  $c(1_T) = R$ .
- (3) (Dedekind-Mertens content formula) For each  $f, g \in T$ , there exists a natural number  $n$  such that  $c(f)^nc(g) = c(f)^{n-1}c(fg)$ .

For more information on content algebras and their examples we refer to [11], [12] and [13]. In [10] Nasehpour gave the definition of a Gaussian  $R$ -algebra as follows: Let  $T$  be an  $R$ -algebra such that  $f \in c(f)T$  for all  $f \in T$ .  $T$  is said to be a Gaussian  $R$ -algebra if  $c(fg) = c(f)c(g)$ , for all  $f, g \in T$ .

**Example 3.11.** ([10]) Let  $T$  be a content  $R$ -algebra such that  $R$  is a Prüfer domain. Since every nonzero finitely generated ideal of  $R$  is a cancellation ideal of  $R$ , the Dedekind-Mertens content formula causes  $T$  to be a Gaussian  $R$ -algebra.

**Theorem 3.12.** *Let  $R$  be a Prüfer domain,  $T$  a content  $R$ -algebra and  $Q$  an ideal of  $R$ . Then  $Q$  is a special 2-absorbing primary ideal of  $R$  if and only if  $QT$  is a special 2-absorbing primary ideal of  $T$ .*

*Proof.* ( $\Rightarrow$ ) Assume that  $Q$  is a special 2-absorbing primary ideal of  $R$ . Let  $fgh \in QT$  for some  $f, g, h \in T$ . Then  $c(fgh) \subseteq Q$ . Since  $R$  is a Prüfer domain and  $T$  is a content  $R$ -algebra, then  $T$  is a Gaussian  $R$ -algebra. Therefore  $c(fgh) = c(f)c(g)c(h) \subseteq Q$ . Since  $Q$  is a special 2-absorbing primary ideal of  $R$ , Theorem 3.5 implies that either  $c(f)c(g) = c(fg) \subseteq Q$  or  $c(f)c(h) = c(fh) \subseteq Q$  or  $c(g)c(h) = c(gh) \subseteq \sqrt{Q}$ . So  $fg \in c(fg)T \subseteq QT$  or  $fh \in c(fh)T \subseteq QT$  or  $gh \in c(gh)T \subseteq \sqrt{QT} \subseteq \sqrt{QT}$ . Consequently  $QT$  is a special 2-absorbing primary ideal of  $T$ . ( $\Leftarrow$ ) Note that since  $T$  is a content  $R$ -algebra,  $QT \cap R = Q$  for every ideal  $Q$  of  $R$ . Now, apply Corollary 2.24(1).  $\square$

The algebra of all polynomials over an arbitrary ring with an arbitrary number of indeterminates is an example of content algebras.

**Corollary 3.13.** *Let  $R$  be a Prüfer domain and  $Q$  be an ideal of  $R$ . Then  $Q$  is a special 2-absorbing primary ideal of  $R$  if and only if  $Q[X]$  is a special 2-absorbing primary ideal of  $R[X]$ .*

**Corollary 3.14.** *Let  $S$  be a multiplicatively closed subset of  $R$  and  $Q$  be a proper ideal of  $R$ . Then the following conditions hold:*

- (1) *If  $Q$  is a special 2-absorbing primary ideal of  $R$  such that  $Q \cap S = \emptyset$ , then  $S^{-1}Q$  is a special 2-absorbing primary ideal of  $S^{-1}R$  with  $2\text{-ord}(S^{-1}Q) \leq 2\text{-ord}(Q)$ .*
- (2) *If  $S^{-1}Q$  is a special 2-absorbing primary ideal of  $S^{-1}R$  and  $S \cap Z_Q(R) = \emptyset$ , then  $Q$  is a special 2-absorbing primary ideal of  $R$  with  $2\text{-ord}(Q) \leq 2\text{-ord}(S^{-1}Q)$ .*

*Proof.* By Proposition 2.27.  $\square$

In view of Theorem 2.31 and its proof, we have the following result.

**Corollary 3.15.** *Let  $R = R_1 \times R_2$ , where  $R_1$  and  $R_2$  are rings with  $1 \neq 0$ . Let  $Q$  be a proper ideal of  $R$ . Then the following conditions are equivalent:*

- (1)  *$Q$  is a special 2-absorbing primary ideal of  $R$ ;*
- (2) *Either  $Q = Q_1 \times R_2$  for some special 2-absorbing primary ideal  $Q_1$  of  $R_1$  or  $Q = R_1 \times Q_2$  for some special 2-absorbing primary ideal  $Q_2$  of  $R_2$  or  $Q = Q_1 \times Q_2$  for some prime ideal  $Q_1$  of  $R_1$  and some prime ideal  $Q_2$  of  $R_2$ .*

**Corollary 3.16.** *Let  $R = R_1 \times R_2$ , where  $R_1$  and  $R_2$  are rings with  $1 \neq 0$ . Suppose that  $Q_1$  is a proper ideal of  $R_1$  and  $Q_2$  is a proper ideal of  $R_2$ . Then  $Q_1 \times Q_2$  is a special 2-absorbing primary ideal of  $R$  if and only if it is a 2-absorbing ideal of  $R$ .*

*Proof.* See Corollary 3.15 and apply [1, Theorem 4.7].  $\square$

**Corollary 3.17.** *Let  $R = R_1 \times R_2 \times \cdots \times R_n$ , where  $2 \leq n < \infty$ , and  $R_1, R_2, \dots, R_n$  are rings with  $1 \neq 0$ . For a proper ideal  $Q$  of  $R$  the following conditions are equivalent:*

- (1)  *$Q$  is a special 2-absorbing primary ideal of  $R$ .*
- (2) *Either  $Q = \times_{t=1}^n Q_t$  such that for some  $k \in \{1, 2, \dots, n\}$ ,  $Q_k$  is a special 2-absorbing primary ideal of  $R_k$ , and  $Q_t = R_t$  for every  $t \in \{1, 2, \dots, n\} \setminus \{k\}$  or  $Q = \times_{t=1}^n Q_t$  such that for some  $k, m \in \{1, 2, \dots, n\}$ ,  $Q_k$  is a prime ideal of  $R_k$ ,  $Q_m$  is a prime ideal of  $R_m$ , and  $Q_t = R_t$  for every  $t \in \{1, 2, \dots, n\} \setminus \{k, m\}$ .*



*Proof.* By Theorem 2.33. □

## REFERENCES

- [1] D. F. Anderson and A. Badawi, On  $n$ -absorbing ideals of commutative rings, *Comm. Algebra* **39** (2011) 1646–1672.
- [2] D. D. Anderson, K. R. Knopp and R. L. Lewin, Ideals generated by powers of elements, *Bull. Austral. Math. Soc.*, **49** (1994) 373–376.
- [3] A. Badawi, On 2-absorbing ideals of commutative rings, *Bull. Austral. Math. Soc.*, **75** (2007), 417–429.
- [4] A. Badawi, Ü. Tekir and E. Yetkin, On 2-absorbing primary ideals in commutative rings, *Bull. Korean Math. Soc.*, **51** (4) (2014), 1163–1173.
- [5] A. Badawi and A. Yousefian Darani, On weakly 2-absorbing ideals of commutative rings, *Houston J. Math.*, **39** (2013), 441–452.
- [6] J. A. Cox and A. J. Hetzel, Uniformly primary ideals, *J. Pure Appl. Algebra*, **212** (2008), 1–8.
- [7] M. Hochster, Criteria for equality of ordinary and symbolic powers of primes, *Math. Z.* **133** (1973) 53–65.
- [8] J. Hukaba, *Commutative rings with zero divisors*, Marcel Dekker, Inc., New York, 1988.
- [9] H. Mostafanasab, E. Yetkin, U. Tekir and A. Yousefian Darani, On 2-absorbing primary submodules of modules over commutative rings, *An. St. Univ. Ovidius Constanta*, (in press)
- [10] P. Nasehpour, On the Anderson-Badawi  $\omega_{R[X]}(I[X]) = \omega_R(I)$  conjecture, arXiv:1401.0459, (2014).
- [11] D. G. Northcott, A generalization of a theorem on the content of polynomials, *Proc. Cambridge Phil. Soc.*, **55** (1959), 282–288.
- [12] J. Ohm and D. E. Rush, Content modules and algebras, *Math. Scand.*, **31** (1972), 49–68.
- [13] D. E. Rush, Content algebras, *Canad. Math. Bull.*, **21** (3) (1978), 329–334.
- [14] R.Y. Sharp, *Steps in commutative algebra*, Second edition, Cambridge University Press, Cambridge, 2000.
- [15] A. Yousefian Darani and F. Soheilnia, 2-absorbing and weakly 2-absorbing submodules, *Thai J. Math.* **9**(3) (2011) 577–584.
- [16] A. Yousefian Darani and F. Soheilnia, On  $n$ -absorbing submodules, *Math. Comm.*, **17** (2012), 547–557.

Hojjat Mostafanasab

Department of Mathematics and Applications,

University of Mohaghegh Ardabili,

P. O. Box 179, Ardabil, Iran.

Email: h.mostafanasab@uma.ac.ir, h.mostafanasab@gmail.com

Ünsal Tekir and Gülşen Ulucak

Department of Mathematics,

Marmara University,

Ziverbey, Goztepe, Istanbul 34722, Turkey.

Email: utekir@marmara.edu.tr, gulsenulucak58@gmail.com